NAG Fortran Library Routine Document

S19ADF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

S19ADF returns a value for the Kelvin function kei x via the routine name.

2 Specification

```
real FUNCTION S19ADF(X, IFAIL)
INTEGER            IFAIL
real               X
```

3 Description

This routine evaluates an approximation to the Kelvin function kei x.

Note: for $x < 0$ the function is undefined, so we need only consider $x \geq 0$.

The routine is based on several Chebyshev expansions:

For $0 \leq x \leq 1$,

$$\text{kei}\ x = -\frac{\pi}{4} f(t) + \frac{x^2}{4}[-g(t)\log x + v(t)]$$

where $f(t)$, $g(t)$ and $v(t)$ are expansions in the variable $t = 2x^4 - 1$;

For $1 < x \leq 3$,

$$\text{kei}\ x = \exp\left(-\frac{9}{8}x\right)u(t)$$

where $u(t)$ is an expansion in the variable $t = x - 2$;

For $x > 3$,

$$\text{kei}\ x = \sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}\left[\left(1 + \frac{1}{x}\right)c(t)\sin\beta + \frac{1}{x}d(t)\cos\beta\right]$$

where $\beta = \frac{x}{\sqrt{2}} + \frac{\pi}{8}$, and $c(t)$ and $d(t)$ are expansions in the variable $t = \frac{6}{x} - 1$.

For $x < 0$, the function is undefined, and hence the routine fails and returns zero.

When x is sufficiently close to zero, the result is computed as

$$\text{kei}\ x = -\frac{\pi}{4} + \left(1 - \gamma - \log\left(\frac{x}{2}\right)\right)\frac{x^2}{4}$$

and when x is even closer to zero simply as

$$\text{kei}\ x = -\frac{\pi}{4}.$$

For large x, kei x is asymptotically given by $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$ and this becomes so small that it cannot be computed without underflow and the routine fails.

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

5 Parameters

1: X – ***real*** *Input*

On entry: the argument x of the function.

Constraint: $X \geq 0$.

2: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL $= 1$

On entry, X is too large, the result underflows. On soft failure, the routine returns zero.

IFAIL $= 2$

On entry, $X < 0$, the function is undefined. On soft failure the routine returns zero.

7 Accuracy

Let E be the absolute error in the result, and δ be the relative error in the argument. If δ is somewhat larger than the machine representation error, then we have:

$$E \simeq \left| \frac{x}{\sqrt{2}} (-\operatorname{ker}_1 x + \operatorname{kei}_1 x) \right| \delta.$$

For small x, errors are attenuated by the function and hence are limited by the ***machine precision***.

For medium and large x, the error behaviour, like the function itself, is oscillatory and hence only absolute accuracy of the function can be maintained. For this range of x, the amplitude of the absolute error decays like $\sqrt{\frac{\pi x}{2}} e^{-x/\sqrt{2}}$, which implies a strong attenuation of error. Eventually, kei x, which is asymptotically given by $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$,becomes so small that it cannot be calculated without causing underflow and therefore the routine returns zero. Note that for large x, the errors are dominated by those of the Fortran intrinsic function EXP.

8 Further Comments

Underflow may occur for a few values of x close to the zeros of kei x, below the limit which causes a failure with IFAIL $= 1$.

9 Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     S19ADF Example Program Text
*     Mark 14 Revised.  NAG Copyright 1989.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
*     .. Local Scalars ..
      real             X, Y
      INTEGER          IFAIL
*     .. External Functions ..
      real             S19ADF
      EXTERNAL         S19ADF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'S19ADF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
      WRITE (NOUT,*)
      WRITE (NOUT,*) '     X            Y        IFAIL'
      WRITE (NOUT,*)
   20 READ (NIN,*,END=40) X
      IFAIL = 1
*
      Y = S19ADF(X,IFAIL)
*
      WRITE (NOUT,99999) X, Y, IFAIL
      GO TO 20
   40 STOP
*
99999 FORMAT (1X,1P,2e12.3,I7)
      END
```

9.2 Program Data

```
S19ADF Example Program Data
             0.0
             0.1
             1.0
             2.5
             5.0
             10.0
             15.0
             1100.0
             -1.0
```

9.3 Program Results

```
 S19ADF Example Program Results

      X           Y        IFAIL

    0.000E+00  -7.854E-01      0
    1.000E-01  -7.769E-01      0
    1.000E+00  -4.950E-01      0
    2.500E+00  -1.107E-01      0
    5.000E+00   1.119E-02      0
    1.000E+01  -3.075E-04      0
    1.500E+01   7.963E-06      0
    1.100E+03   0.000E+00      1
   -1.000E+00   0.000E+00      2
```